A REMARK ON THE TENSOR PRODUCTS OF SC-RECIPROCITY SHEAVES

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Abstract. We give a description of the tensor product of SC-reciprocity presheaves with transfers in terms of K-group of geometric type. By this description and concrete computations, we have a description of Chow group of 0-cycles with modulus of products of modulus pairs of the form $(\mathbb{P}^1, \mathfrak{m})$. We also apply our result to show that the tensor product of $\mathbb{G}_a$ and $\mathbb{G}_m$ is isomorphic to the sheaf Ω^1 of Kähler differential forms as reciprocity sheaves over characteristic zero.

1. Introduction

In [8], Kahn, Saito and Yamazaki have defined "reciprocity presheaves with transfers" as a generalization of homotopy invariant presheaves with transfers. But there are some problem on the category **Rec** of reciprocity presheaves with transfers. One of the problems is a tensor structure in **Rec**; we don't know whether **Rec** has a tensor structure. Recently, in [9] Kahn, Saito and Yamazaki have constructed a triangulated category of "motive with modulus", and introduced "SC-reciprocity presheaves with transfers". The category **RSC** of SC-reciprocity presheaves with transfers is a subcategory of **Rec**, and **RSC** has the tensor structure. In this paper, we study the tensor structure in **RSC**.

Kahn and Yamazaki [7] give a description of the tensor product of homotopy invariant sheaves in terms of a certain K-group. For non-homotopy invariant sheaves, Ivorra and Rülling [5] have defined T-functor group for their reciprocity functors. For example, homotopy invariant sheaves, the additive group $\mathbb{G}_a$ and the absolute Kähler differential Ω^1_k give rise to reciprocity functors. The T-functor group of homotopy invariant sheaves coincides with the tensor product of those sheaves, i.e. Kahn-Yamazaki's K-group. One of the interesting computation is that Ω^1_k is isomorphic to T-functor group associated to $\mathbb{G}_a$ and $\mathbb{G}_m$ (cf. Hiranouchi [3] has also computed it by his K-group).

In this paper, we define a K-group $K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1 \dots, F_n)$ of geometric type for SC-reciprocity sheaves $F_1, \dots, F_n$, similarly to Kahn-Yamazaki's K-group of geometric type by adding a "modulus condition" (Definition 3.2). The following theorem is our main result which gives a description of a tensor product of SC-reciprocity presheaves with transfers in terms of our K-group.

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Theorem 1.1 (Theorem 3.1)**.** *Let $F_1, \dots, F_n$ be SC-reciprocity presheaves with transfers. Then there is an isomorphism*

$$(F_1 \otimes_{\mathbf{RSC}} \cdots \otimes_{\mathbf{RSC}} F_n)(k) \simeq K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1 \dots, F_n)$$

We compute our K-group for $\mathbb{G}_m$ and $\mathbb{G}_a$ (Proposition 4.2, Proposition 4.7), which recovers some computations in [3, 5, 7]. This computation tells us that the Kähler differential r-forms Ω^r_k is the value on $\mathrm{Spec}(k)$ of the tensor product $\mathbb{G}_a$ and r-copies of $\mathbb{G}_m$ in **RSC**. Our K-group for $\mathbb{G}_a$ and $\mathbb{G}_a$ behaves different from T-functor group (Proposition 4.8 and Corollary 5.4). As application, we give a description of Chow group of 0-cycles with modulus for products of curves (see §5.1). Combining those computations, we have

Corollary 1.2 (Corollary 5.8)**.** *There is an isomorphism of reciprocity Zariski sheaves*

$$\Omega^1 \simeq (\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}}.$$

After a review of some materials about SC-reciprocity sheaves and its tensor product in §2, we prove Theorem 1.1 in §3. In §4, we compute our K-group for $\mathbb{G}_m$ and $\mathbb{G}_a$. In §5, we apply Theorem 1.1 and the computations in §4 to 0-cycles with modulus and prove Corollary 1.2 as consequence.

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Notation and conventions. Let k be a perfect base field. All schemes over k are separated schemes of finite type over k. We write **Cor** for Voevodsky's category of finite correspondences. We write **PST** for the category of presheaves with transfers and **HI** for the full subcategory of **PST** of homotopy($\mathbb{A}^1$-) invariant presheaves.

2. Category of SC-reciprocity sheaves and the tensor product

We recall the definition of SC-reciprocity sheaves and its tensor product from [9]. We start from recalling a definition of proper modulus pairs.

2.1. Proper modulus pairs and admissible correspondences.

Definition 2.1. (1) A modulus pair $M = (\overline{M}, M^\infty)$ consists of a separated scheme $\overline{M}$ of finite type over k and an effective Cartier divisor M^∞ on $\overline{M}$ such that $\overline{M}$ is locally integral and the dense open subset $M^o := \overline{M} \setminus |M^\infty|$ is smooth over k. A modulus pair M is called *proper* if $\overline{M}$ is proper over k.

We write $\overline{\square}$ for a modulus pair $(\mathbb{P}^1, 1)$

(2) Let M, N be modulus pairs and let Z be an elementary correspondence from M^o to N^o. Let $\overline{Z}^N$ be the normalization of the closure of Z in $\overline{M} \times \overline{N}$ and let p_M, p_N be the canonical morphism from $\overline{Z}^N$ to $\overline{M}, \overline{N}$ respectively. We say Z is *adimissible* if $p_M^*(M^\infty) \geq p_N^*(N^\infty)$. An element of $\mathbf{Cor}(M^o, N^o)$ is called *admissible* if all of its irreducible components are admissible.

(3) Let **MCor** be the category of proper modulus pair over k, whose objects are proper modulus pairs and morphisms are admissible correspondences.

(4) Let M, N be modulus pairs. Then we define a tensor product $L = M \otimes N$ by

$$\overline{L} = \overline{M} \times \overline{N}, \quad L^\infty = M^\infty \times \overline{N} + \overline{M} \times N^\infty.$$

The pair $(\mathrm{Spec}(k), \phi)$ is the unit with this tensor product.

Remark 2.2. Let C be a proper smooth curve over k and $\mathfrak{m}$ be an effective Cartier divisor on C. We write

$$G(C, \mathfrak{m}) := \bigcap_{x \in \mathfrak{m}} \mathrm{Ker}\big(\mathcal{O}^\times_{\overline{C},x} \to \mathcal{O}^\times_{\mathfrak{m},x}\big) = \varinjlim_{\mathfrak{m} \subset U \subset \overline{C}} \Gamma(U, \ker(\mathcal{O}^\times_{\overline{C}} \to \mathcal{O}^\times_{\mathfrak{m}})).$$

Then a transpose Γ_f^t of the graph of an element $f \in G(C, \mathfrak{m}) \setminus \{1\}$ is an admissible correspondence in $\mathbf{MCor}(\overline{\square}, (C, \mathfrak{m}))$.

2.2. Modulus presheaves with transfers.

Definition 2.3 ([9, Definition 2.1])**.** Let **MPST** be the category of additive presheaves on **MCor**, which we call the category of modulus presheaves with transfers.

For any proper modulus pair M, we write $\mathbb{Z}_{\mathrm{tr}}(M) \in \mathbf{MPST}$ for the representative modulus presheaf with transfers.

We have a forgetful functor $\omega : \mathbf{MCor} \to \mathbf{Cor}$ which sends M to M^o. By [9, Proposition 2.2], we have a monoidal exact functor $\omega_! : \mathbf{MPST} \to \mathbf{PST}$ which is left adjoint to ω^*.

Let $F, G \in \mathbf{MPST}$. There are exact sequences

$$\bigoplus_j \mathbb{Z}_{\mathrm{tr}}(N_j) \longrightarrow \bigoplus_i \mathbb{Z}_{\mathrm{tr}}(M_i) \longrightarrow F \longrightarrow 0$$

$$\bigoplus_{j'} \mathbb{Z}_{\mathrm{tr}}(N_{j'}) \longrightarrow \bigoplus_{i'} \mathbb{Z}_{\mathrm{tr}}(M_{i'}) \longrightarrow G \longrightarrow 0.$$

Then the tensor product $F \otimes_{\mathbf{MPST}} G$ of F and G is defined as

$$F \otimes_{\mathbf{MPST}} G$$
$$:= \mathrm{Coker}\left(\bigoplus_{j,i'} \mathbb{Z}_{\mathrm{tr}}(N_j \otimes M'_{i'}) \oplus \bigoplus_{i,j'} \mathbb{Z}_{\mathrm{tr}}(M_i \otimes N_{j'}) \longrightarrow \bigoplus_{i,i'} \mathbb{Z}_{\mathrm{tr}}(M_i \otimes M_{i'})\right).$$

From the definition, we have

$$(F \otimes_{\mathbf{MPST}} G)((\mathrm{Spec}(k), \phi)) \simeq (\omega_! F \otimes_{\mathbf{PST}} \omega_! G)(\mathrm{Spec}(k)).$$

Throughout the paper, we write just $F(k)$ for $F(\mathrm{Spec}(k), \phi)$.

2.3. $\overline{\square}$-invariant modulus presheaves.

Definition 2.4 ([9, Definition 7.1])**.** (1) Let F be a modulus presheaf with transfers. We say F is $\overline{\square}$-invariant if the projection $M \times \overline{\square} \to M$ induces an isomorphism $p^* : F(M) \simeq F(M \times \overline{\square})$ for any proper modulus pair M.

Let **CI** be the full subcategory of **MPST** of $\overline{\square}$-invariant modulus presheaves with transfers.

(2) For $F \in \mathbf{MPST}$, we define $H_0(F) \in \mathbf{CI}$ as

$$H_0(F) := \mathrm{Coker}\left(\delta_0^* - \delta_\infty^* : \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}), F) \to F\right).$$

Here $\delta_0, \delta_\infty : \mathrm{Spec}(k) \to \overline{\square}$ be the zero and ∞-sections respectively.

Let $F, G \in \mathbf{CI}$. Then the tensor product $F \otimes_{\mathbf{CI}} G$ of F and G in $\mathbf{CI}$ is defined as

$$F \otimes_{\mathbf{CI}} G := H_0(F \otimes_{\mathbf{MPST}} G)$$

and hence

$$(F \otimes_{\mathbf{CI}} G)(k) = \mathrm{Coker}\left((F \otimes_{\mathbf{MPST}} G)(\overline{\square}) \longrightarrow (F \otimes_{\mathbf{MPST}} G)(k)\right).$$

2.4. SC-reciprocity presheaves with transfers.

Definition 2.5. For any proper modulus pair M, we define $h_0(M) \in \mathbf{PST}$ as

$$\begin{aligned} h_0(M) : &= \omega_!(H_0(\mathbb{Z}_{\mathrm{tr}}(M))) \\ &= \mathrm{Coker}\left(\delta_0^* - \delta_\infty^* : \omega_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}), \mathbb{Z}_{\mathrm{tr}}(M)) \to \mathbb{Z}_{\mathrm{tr}}(M^o)\right). \end{aligned}$$

Remark 2.6. For a proper modulus pair $M = (\overline{X}, D)$, the group $h_0(M)(\mathrm{Spec}(k))$ is the Chow group $\mathrm{CH}_0(M) := \mathrm{CH}_0(\overline{X}, D)$ of 0-cycles with modulus for $M = (\overline{X}, D)$. For the definition of (higher-)Chow group with modulus, refer to [1, 8, 6].

Definition 2.7 (see [9, Definition 7.6])**.** (1) Let $F \in \mathbf{PST}$. Let X be a smooth scheme over k and $a \in F(X) = \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)$. A proper modulus pair M with $M^o = X$ is said to be a *SC-modulus* for a if a factors through $\mathbb{Z}_{\mathrm{tr}}(X) \to h_0(M)$.

(2) We say $F \in \mathbf{PST}$ has *SC-reciprocity* if for any smooth X and any $a \in F(X)$, there is an SC-modulus M for a.

We write $\mathbf{RSC}$ for the full subcategory of $\mathbf{PST}$ of SC-reciprocity presheaves with transfers.

Remark 2.8. (1) By [9, Proposition 7.7], if $F \in \mathbf{MPST}$ is $\overline{\square}$-invariant then $\omega_! F \in \mathbf{PST}$ has SC-reciprocity. Thus we have an exact functor $\omega_{\mathbf{CI}} : \mathbf{CI} \to \mathbf{RSC}$ induced by $\omega_!$. Furthermore, we have a left exact functor $\omega^{\mathbf{CI}} : \mathbf{RSC} \to \mathbf{CI}$ which is right adjoint to $\omega_{\mathbf{CI}}$, and the adjunction map $\omega_{\mathbf{CI}}\omega^{\mathbf{CI}} \Rightarrow \mathrm{id}_{\mathbf{RSC}}$ is an isomorphism (see [9, Proposition 7.22]).

(2) By [9, Theorem 7.10], SC-reciprocity implies reciprocity in the sense of [8]. Thus $\mathbf{RSC}$ is a subcategory of the category $\mathbf{Rec}$ of reciprocity presheeaves with transfers.

Let $F, G \in \mathbf{RSC}$. The tensor product $F \otimes_{\mathbf{RSC}} G$ of F and G is defined as

$$F \otimes_{\mathbf{RSC}} G := \omega_{\mathbf{CI}}(\omega^{\mathbf{CI}} F \otimes_{\mathbf{CI}} \omega^{\mathbf{CI}} G).$$

Here $\omega_{\mathbf{CI}}, \omega^{\mathbf{CI}}$ are the adjoint functors between $\mathbf{RSC}$ and $\mathbf{CI}$ (see Remark 2.8 (1)). Thus we have

$$\begin{aligned} (F \otimes_{\mathbf{RSC}} G)(k) &= \omega_{\mathbf{CI}}(\omega^{\mathbf{CI}} F \otimes_{\mathbf{CI}} \omega^{\mathbf{CI}} G)(k) \\ &= (\omega^{\mathbf{CI}} F \otimes_{\mathbf{CI}} \omega^{\mathbf{CI}} G)(k). \end{aligned}$$

3. K-group of geometric type

In this section, following [7], we define a K-group $K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1 \dots, F_n)$ of geometric type for SC-reciprocity sheaves $F_1, \dots, F_n$ and we prove

Theorem 3.1. *Let $F_1, \dots, F_n$ be SC-reciprocity presheaves with transfers. Then there is an isomorphism*

$$(F_1 \otimes_{\mathbf{RSC}} \cdots \otimes_{\mathbf{RSC}} F_n)(k) \simeq K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1 \dots, F_n)$$

3.1. Definition of K-group of geometric type.

Definition 3.2 (K-group of geometric type)**.** Let $F_1, \dots, F_n$ be SC-reciprocity presheaves with transfers.

(i) A *relation datum of geometric type with summational modulus* for $\{F_i\}_i$ is a collection $((C, \mathfrak{m}), f, \{g_i\}_i)$ of the following objects:
  (1) $(C, \mathfrak{m})$ is a proper modulus pair with a proper smooth curve C over k. We write $C^o = C \setminus \mathfrak{m}$ for the open complement.
  (2) f is a function in $G(C, \mathfrak{m})$ (cf. Remark 2.2).
  (3) g_i is of element $F_i(C \setminus \mathfrak{m}_i)$ having SC-modulus $(C, \mathfrak{m}_i)$, i.e. g_i is an element in $\mathrm{Hom}_{\mathbf{PST}}(h_0(C, \mathfrak{m}_i), F_i)$.
  (4) $\mathfrak{m}$ satisfies an inequality $\mathfrak{m} \geq \sum_{i=1}^{n} \mathfrak{m}_i$.

(ii) We define K-group $K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1 \dots, F_n)$ of geometric type attached to $\{F_i\}_i$ with respect to summational modulus as follows:

$$K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1 \dots, F_n) := \big(F_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} F_n\big)(k)/\mathcal{R}.$$

Here $\mathcal{R}$ is the subgroup generated by the elements of the following form

$$\sum_{c \in C^o} v_c(f) \mathrm{Tr}_{k(c)/k}(g_1(c) \otimes \cdots \otimes g_n(c))$$

for all relation datum $((C, \mathfrak{m}), f, \{g_i\}_i)$ of geometric type with summational modulus for $\{F_i\}_i$.

Remark 3.3. One can similarly define K-group $K^{\mathrm{geo}}_{\max}(k; F_1, \dots, F_n)$ of geometric type with *maximal modulus* by just replacing the inequality in Definition 3.2 (i)(4) by $\mathfrak{m} \geq \max\{\mathfrak{m}_i\}$. One can easily check that $K^{\mathrm{geo}}_{\max}$ coincides with K-group of geometric type of Kahn–Yamazaki [7] when all F_i are homotopy invariant. Also, $K^{\mathrm{geo}}_{\max}$ coincides with LT-functor in [5, Definition 4.2.3]. But it is not obvious that $K^{\mathrm{geo}}_{\max}$ coincides with T-functor of Ivorra-Rülling [5].

Since $\sum_{i=1}^{n} \mathfrak{m}_i \geq \max \mathfrak{m}_i$, we have a canonical surjection

$$\eta : K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1, \dots, F_n) \longrightarrow K^{\mathrm{geo}}_{\max}(k; F_1, \dots, F_n).$$

Question. *When is the map η bijective?*

We will give an answer to the question in some special cases (see §4).

3.2. Lemmas.

Let F be in $\mathbf{MPST}$. Suppose that we are given the following data:

(a) a proper modulus pair $(C, \mathfrak{m})$ with a proper smooth curve C,
(b) a surjective morphism $f : C \to \mathbb{P}^1$ such that $\mathfrak{m} \leq f^*([1])$,

(c) a morphism $\alpha : \mathbb{Z}_{\mathrm{tr}}(C, \mathfrak{m}) \to F$ in **MPST**.

To such data $((C, \mathfrak{m}), f, \alpha)$, we associate the following element in $F(k)$;

$$\alpha(\mathrm{div}(f)) \in F(k). \tag{3.1}$$

This element can be written as the following form.

$$\alpha(\mathrm{div}(f)) = \sum_{c \in C^o} v_c(f) \mathrm{Tr}_{k(c)/k}(\alpha(c)) \in F(k).$$

Lemma 3.4. *Let $F \in$ **MPST**. We define $F(k)_{\mathrm{m-rat}}$ to be the subgroup of $F(k)$ generated by elements* (3.1) *for all triples $((C, \mathfrak{m}), f, \alpha)$. Then we have*

$$H_0(F)(k) = F(k)/F(k)_{\mathrm{m-rat}}.$$

Proof. By definition, we have

$$H_0(F)(k) = \mathrm{Coker}\left(i_0^* - i_\infty^* : F(\overline{\square}) \to F(k)\right).$$

For a triple $((C, \mathfrak{m}), f, \alpha)$, let M denote the proper modulus pair $(C, \mathfrak{m})$. By condition on f, the graph Γ_f of f belongs to $\mathbf{MCor}(\overline{\square}, M)$, and hence we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_{\mathrm{tr}}(M)(\overline{\square}) \arrow[r, "\alpha"] \arrow[d, "i_0^* - i_\infty^*"] & F(\overline{\square}) \arrow[d, "i_0^* - i_\infty^*"] \\
\mathbb{Z}_{\mathrm{tr}}(M)(k) \arrow[r] & F(k).
\end{tikzcd}$$

The image of Γ_f in $\mathbb{Z}_{\mathrm{tr}}(M)(k)$ is $\mathrm{div}(f)$, which shows that $F(k)_{\mathrm{m-rat}}$ is zero in $h_0(F)(k)$.

Conversely, for any element $\alpha \in F(\overline{\square})$, the element $(i_0^* - i_\infty^*)(\alpha)$ coincides with (3.1) for the triple $(\overline{\square}, \mathrm{id}, \alpha)$. $\square$

Lemma 3.5. *Let $F_1, \ldots, F_n \in$ **MPST** and let $F = F_1 \otimes_{\mathbf{MPST}} \cdots \otimes_{\mathbf{MPST}} F_n$. Let $(C, \mathfrak{m})$ be a proper modulus pair with a smooth curve C. Then $\alpha \in F(C, \mathfrak{m})$ is the sum of elements of the form*

$$\mathrm{Tr}_h(g_1 \otimes \cdots \otimes g_n)$$

where $(C', \mathfrak{n})$ is a modulus pair with a proper smooth curve C', $h : C' \to C$ is a finite surjective morphism, $g_i \in F_i(C', \mathfrak{n})$ for $i = 1, \ldots, n$, $\mathrm{Tr}_h : F(C', \mathfrak{n}) \to F(C, \mathfrak{m})$ is the transfer with respect to h.

Proof. We may assume that all F_i are representative $\mathbb{Z}_{\mathrm{tr}}(M_i)$. In this case, we have $F = \mathbb{Z}_{\mathrm{tr}}(M)$ with $M = M_1 \otimes \cdots \otimes M_n$.

Let $Z \in F(C, \mathfrak{m}) = \mathbf{MCor}((C, \mathfrak{m}), M)$ be an admissible elementary finite correspondence. Let $\overline{Z}$ be the closure of Z in $C \times \overline{M}$. Let C' be the normalization of $\overline{Z}$ and let $\mathfrak{n}$ be the pull-back of $\mathfrak{m}$ along a surjective morphism $h : C' \to \overline{Z} \to C$. Put $N = (C', \mathfrak{n})$. For $i = 1, \ldots, n$, by the definition of $\mathfrak{n}$, the graph of $C'^o \to Z \to M_i^o$ gives an element $g_i \in \mathbf{MCor}(N, M_i)$, since Z is admissible. Let Γ_h^t be the transpose of the graph of h, which belongs to $\mathbf{MCor}((C, \mathfrak{m}), N)$. If we take $g = g_1 \otimes \cdots \otimes g_n \in \mathbf{MCor}(N, M)$, then by definition we have $\mathrm{Tr}_h(g) = Z \in \mathbf{MCor}((C, \mathfrak{m}), M)$. This completes the proof. $\square$

3.3. **Proof of Theorem 3.1.** From the definition of $\otimes_{\mathbf{RSC}}$ (see §2.4),

$$(F_1 \otimes_{\mathbf{RSC}} \cdots \otimes_{\mathbf{RSC}} F_n)(k) = H_0(\omega^{\mathbf{CI}} F_1 \otimes_{\mathbf{MPST}} \cdots \otimes_{\mathbf{MPST}} \omega^{\mathbf{CI}} F_n)(k).$$

By Lemma 3.4, Lemma 3.5 and the definition of $K^{\mathrm{geo}}_{\mathrm{sum}}$ (see Definition 3.2), we have

$$H_0(\omega^{\mathbf{CI}} F_1 \otimes_{\mathbf{MPST}} \cdots \otimes_{\mathbf{MPST}} \omega^{\mathbf{CI}} F_n)(k) = K^{\mathrm{geo}}_{\mathrm{sum}}(k; F_1, \ldots, F_n).$$

4. Some computations

Remark 4.1. The same proof in [8, §4] works for showing SC-reciprocity for commutative algebraic groups. This implies the following: let $M = (C, \mathfrak{m})$ be a proper modulus pair with a proper smooth curve C. Then the Zariski sheafification $h_0^{\mathrm{Zar}}(M)$ of $h_0(M)$ has SC-reciprocity, since we have the following isomorphism ([8, Proposition 9.4.1])

$$h_0^{\mathrm{Zar}}(M)^0 \simeq J_M$$

where $h_0^{\mathrm{Zar}}(M)^0$ is the kernel of the degree map $h_0^{\mathrm{Zar}}(M) \to \mathbb{Z}$ and J_M is the Rosenlicht-Serre generalized Jacobian for $M = (C, \mathfrak{m})$.

4.1. **K-group of copies of $\mathbb{G}_m$.**

Proposition 4.2. *We have an isomorphism*

$$(\mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}})(k) \simeq K^{\mathrm{geo}}_{\mathrm{sum}}(k; \mathbb{G}_m, \ldots, \mathbb{G}_m) \simeq K^M_r(k).$$

Proof. Since we know $K^{\mathrm{geo}}_{\max}(k; \mathbb{G}_m, \ldots, \mathbb{G}_m) \simeq K^M_r(k)$ and we have a canonical surjection

$$K^{\mathrm{geo}}_{\mathrm{sum}}(k; \mathbb{G}_m, \ldots, \mathbb{G}_m) \longrightarrow K^{\mathrm{geo}}_{\max}(k; \mathbb{G}_m, \ldots, \mathbb{G}_m),$$

it suffices to show that the elements of the form

$$a \otimes (1-a) \otimes b_3 \otimes \cdots \otimes b_r \qquad (a \in k \setminus \{0,1\}, b_i \in k^*)$$

are already zero in $K^{\mathrm{geo}}_{\mathrm{sum}}(k; \mathbb{G}_m, \ldots, \mathbb{G}_m)$.

Given $a \in k \setminus \{0,1\}$ and $b_i \in k^*$ for $i = 3, \ldots, r$, we consider the following functions

$$\begin{aligned} g_1 &= t \in \mathbb{G}_m(\mathbb{P}^1 \setminus \{0, \infty\}), \\ g_2 &= 1 - t \in \mathbb{G}_m(\mathbb{P}^1 \setminus \{0, 1, \infty\}), \\ g_i &= b_i \in \mathbb{G}_m(\mathbb{P}^1) \quad (i = 3, \ldots, r) \end{aligned}$$

and the rational function f on $\mathbb{P}^1$,

$$f = \frac{t^6 - (a^6+1)t^4 + (a^6+1)t^2 - a^6}{t^6 - a^6}.$$

Then one can easily check that g_1, g_2 and g_i have SC-modulus $(\mathbb{P}^1, [0] + [\infty]), (\mathbb{P}^1, [0] + [1] + [\infty])$ and $(\mathbb{P}^1, \phi)$ respectively, and that f belongs to $G(\mathbb{P}^1, 2[0] + [1] + 2[\infty])$. Therefore the element

$$\alpha = \sum_{c \in \mathbb{P}^1 \setminus \{0,1,\infty\}} v_c(f) \mathrm{Tr}_{k(c)/k}(g_1(c) \otimes g_2(c) \otimes b_3 \otimes \cdots \otimes b_r)$$

is zero in our $K^{\mathrm{geo}}_{\mathrm{sum}}$. We compute 6α over $E = k(\zeta)$ where ζ is a primitive 12-th root of unity. We write

$$\{(g_1(c), g_2(c), \boldsymbol{b}\}_{k(c)} = g_1(c) \otimes g_2(c) \otimes b_3 \otimes \cdots \otimes b_r \qquad \text{in} \quad K^{\mathrm{geo}}_{\mathrm{sum}}$$

where $\boldsymbol{b}$ denotes $b_3, \dots, b_r$. In $K^{\mathrm{geo}}_{\mathrm{sum}}(E; \mathbb{G}_m, \dots, \mathbb{G}_m)$, we have

$$\begin{aligned} 0 = 6\alpha = 6\{a^3, 1-a^3, \boldsymbol{b}\}_E &+ 6\{-a^3, 1+a^3, \boldsymbol{b}\}_E \\ &+ 6\{\zeta, 1-\zeta, \boldsymbol{b}\}_E + 6\{-\zeta, 1+\zeta, \boldsymbol{b}\}_E \\ &+ 6\{\zeta^5, 1-\zeta^5, \boldsymbol{b}\}_E + 6\{-\zeta^5, 1+\zeta^5, \boldsymbol{b}\}_E \\ &- 6\{a, 1-a, \boldsymbol{b}\}_E - 6\{\zeta^4 a, 1-\zeta^4 a, \boldsymbol{b}\}_E - 6\{\zeta^8 a, 1-\zeta^8 a, \boldsymbol{b}\}_E \\ &- 6\{a, 1+a, \boldsymbol{b}\}_E - 6\{-\zeta^4 a, 1+\zeta^4 a, \boldsymbol{b}\}_E - 6\{-\zeta^8 a, 1+\zeta^8 a, \boldsymbol{b}\}_E \\ = 2\{a^6, 1-a^6, \boldsymbol{b}\}_E. \end{aligned}$$

By taking the norm, we have

$$8\{a^6, 1-a^6, \boldsymbol{b}\}_k = 0 \qquad \text{in} \quad K^{\mathrm{geo}}_{\mathrm{sum}}(k; \mathbb{G}_m, \dots, \mathbb{G}_m)$$

By passing to the extension $k(\sqrt[6]{a})$ and taking the norm, we have

$$48\{a, 1-a, \boldsymbol{b}\}_k = 0.$$

Then $\{a, 1-a, \boldsymbol{b}\}_k = 0$ follows from the lemma below, which completes the proof. □

Lemma 4.3 (See [10, Lemma 5.8])**.** *Assume that there exists an integer $n > 0$ such that $n\{a, 1-a\}_E = 0$ for any finite extension E of k and for any $a \in E \setminus \{0, 1\}$. Then $\{a, 1-a\}_k = 0$ for any $a \in k \setminus \{0, 1\}$.*

Definition 4.4 ([7, Example 10.2 (b)])**.** A homotopy invariant presheaf F with transfers is said to be *birational* if for any smooth k-variety X and any dense open subset U of X, the restriction map $F(X) \to F(U)$ is bijective.

Corollary 4.5. *Let F be a birational homotopy invariant presheaf with transfers. Then there is an isomorphism*

$$(F \otimes_{\mathbf{RSC}} \mathbb{G}_m \otimes_{\mathbf{RSC}} \cdots \otimes_{\mathbf{RSC}} \mathbb{G}_m)(k) \simeq (F \otimes_{\mathbf{HI}} \mathbb{G}_m \otimes_{\mathbf{HI}} \cdots \otimes_{\mathbf{HI}} \mathbb{G}_m)(k).$$

Proof. By Theorem 3.1 and a result of Kahn-Yamazaki [7], our task is to prove an isomorphism

$$K^{\mathrm{geo}}_{\mathrm{sum}}(k; F, \mathbb{G}_m, \dots, \mathbb{G}_m) \simeq K^{\mathrm{geo}}_{\mathrm{max}}(k; F, \mathbb{G}_m, \dots, \mathbb{G}_m).$$

Since F is birational, we have $F(C^o) = F(C)$ for a proper smooth curve C and an open subset C^o of C. Thus every elements of $F(C^o)$ has empty modulus. Therefore, by the definition of our $K^{\mathrm{geo}}_{\mathrm{sum}}$ and $K^{\mathrm{geo}}_{\mathrm{max}}$, we have

$$K^{\mathrm{geo}}_{\mathrm{sum}}(k; F, \mathbb{G}_m^{\otimes^r_{\mathbf{HI}}}) \simeq K^{\mathrm{geo}}_{\mathrm{max}}(k; F, \mathbb{G}_m^{\otimes^r_{\mathbf{HI}}}) \simeq K^{\mathrm{geo}}_{\mathrm{max}}(k; F, \mathbb{G}_m, \dots, \mathbb{G}_m).$$

Here $\mathbb{G}_m^{\otimes^r_{\mathbf{HI}}}$ denotes the tensor product of r-copies of $\mathbb{G}_m$ in $\mathbf{HI}$.

On the other hand, let $U(\mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}})$ be the kernel of the canonical surjection $\mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}} \longrightarrow \mathbb{G}_m^{\otimes^r_{\mathbf{HI}}}$. Then we have an exact sequence

$$K^{\mathrm{geo}}_{\mathrm{sum}}(k; F, U(\mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}})) \longrightarrow K^{\mathrm{geo}}_{\mathrm{sum}}(k; F, \mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}}) \longrightarrow K^{\mathrm{geo}}_{\mathrm{sum}}(k; F, \mathbb{G}_m^{\otimes^r_{\mathbf{HI}}}).$$

By Proposition 4.2, we have

$$U(\mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}})(k) = 0,$$

and hence

$$K^{\mathrm{geo}}_{\mathrm{sum}}(k; F, U(\mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}})) = 0.$$

This completes the proof. □

Remark 4.6. In general, we don't know that for homotopy invariant sheaves, the SC-reciprocity tensor product $\otimes_{\mathbf{RSC}}$ of homotopy invariant sheaves coincides with the homotopy invariant tensor product $\otimes_{\mathbf{HI}}$. This question relates the question on $K^{\mathrm{geo}}_{\mathrm{sum}}$ and $K^{\mathrm{geo}}_{\mathrm{max}}$ in Remark 3.3.

4.2. ***K*-group of $\mathbb{G}_a$ and copies of $\mathbb{G}_m$.**

Proposition 4.7. *Assume that $ch(k) = 0$. We have an isomorphism*

$$(\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m^{\otimes^r_{\mathbf{RSC}}})(k) \simeq K^{\mathrm{geo}}_{\mathrm{sum}}(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m) \simeq \Omega^r_k.$$

Proof. Since the same computation of Ivorra-Rülling [5] works for $K^{\mathrm{geo}}_{\mathrm{max}}$, in case $ch(k) = 0$, we have an isomorphism

$$K^{\mathrm{geo}}_{\mathrm{max}}(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m) \simeq \Omega^r_k.$$

Hence, it suffices to show that the elements of the forms

(i) $\{a, a, b_3, \dots, b_r\} + \{1 - a, 1 - a, b_3, \dots, b_r\}$ $\quad (a \in k \setminus \{0, 1\}, b \in k^*)$
(ii) $\{a, b_1, \dots, b_r\}$ $\quad (a \in k \setminus \{0, 1\}, b \in k^*$ with $b_i = b_j$ for some $i < j$)

are zero in $K^{\mathrm{geo}}_{\mathrm{sum}}(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m)$.

(i) If a is a root of $t^2 - t + 1 = 0$, one easily shows that

$$\{a, a, \boldsymbol{b}\} + \{1 - a, 1 - a, \boldsymbol{b}\} = 0 \qquad \text{in } (\mathbb{G}_a \overset{M}{\otimes} \mathbb{G}_m \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathbb{G}_m)(k)$$

Here $\boldsymbol{b}$ denotes $b_3, \dots, b_r$. Given $a \in k \setminus \{0, 1\}$ with $a^2 - a + 1 \neq 0$ and $b_i \in k^*$ for $i = 3, \dots, r$, we consider the following functions

$$\begin{aligned} &g_1 = t \in \mathbb{G}_a(\mathbb{P}^1 \setminus \{\infty\}), \\ &g_2 = t \in \mathbb{G}_m(\mathbb{P}^1 \setminus \{0, \infty\}), \\ &g_i = b_i \in \mathbb{G}_m(\mathbb{P}^1) \quad (i = 3, \dots, r) \end{aligned}$$

and the rational function f on $\mathbb{P}^1$,

$$f = \frac{(t - a)(t - (1 - a))(t + 1)(t^2 + a^2 - a + 1)}{t^5 - a(1 - a)(a - a^2 - 1)}.$$

Then one can easily check that g_1, g_2 and g_i have SC-modulus $(\mathbb{P}^1, 2[\infty])$, $(\mathbb{P}^1, [0] + [\infty])$ and $(\mathbb{P}^1, \phi)$ respectively, and that f belongs to $G(\mathbb{P}^1, [0] + 3[\infty])$. Therefore the element

$$\sum_{c \in \mathbb{P}^1 \setminus \{0,1,\infty\}} v_c(f) \mathrm{Tr}_{k(c)/k} \{g_1(c), g_2(c), \boldsymbol{b}\}_{k(c)}$$

is zero in our $K^{\mathrm{geo}}_{\mathrm{sum}}$. We write α, β for a root of $t^2 + a^2 - a + 1 = 0$ and $t^5 - a(1 - a)(a - a^2 - 1) = 0$ respectively. In $K^{\mathrm{geo}}_{\mathrm{sum}}(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m)$, we

have

$$
\begin{aligned}
0 &= \sum_{c\in\mathbb{P}^1\setminus\{0,1,\infty\}} v_c(f)\mathrm{Tr}_{k(c)/k}\{g_1(c),g_2(c),\boldsymbol{b}\}_{k(c)}\\
&= \{a,a,\boldsymbol{b}\}_k + \{1-a,1-a,\boldsymbol{b}\}_k + \{-1,-1,\boldsymbol{b}\}_k + \mathrm{Tr}_{k(\alpha)/k}\{\alpha,\alpha,\boldsymbol{b}\}_{k(\alpha)} + \mathrm{Tr}_{k(\beta)/k}\{\beta,\beta,\boldsymbol{b}\}_{k(\beta)}\\
&= \{a,a,\boldsymbol{b}\}_k + \{1-a,1-a,\boldsymbol{b}\}_k + \mathrm{Tr}_{k(\alpha)/k}\{\frac{\alpha}{2},\alpha^2,\boldsymbol{b}\}_{k(\alpha)} + \mathrm{Tr}_{k(\beta)/k}\{\frac{\beta}{5},\beta^5,\boldsymbol{b}\}_{k(\beta)}\\
&= \{a,a,\boldsymbol{b}\}_k + \{1-a,1-a,\boldsymbol{b}\}_k\\
&\qquad + \{\mathrm{Tr}_{k(\alpha)/k}\left(\frac{\alpha}{2}\right),1+a-a^2,\boldsymbol{b}\}_k + \{\mathrm{Tr}_{k(\beta)/k}\left(\frac{\beta}{5}\right),a(1-a)(a-a^2-1),\boldsymbol{b}\}_k\\
&= \{a,a,\boldsymbol{b}\}_k + \{1-a,1-a,\boldsymbol{b}\}_k.
\end{aligned}
$$

(ii) We know that the element of the form

$$\{b_1,\dots,b_r\}_k \qquad \text{with } b_i = b_j \text{ for some } i<j$$

is 2-torsion in $K^{\mathrm{geo}}_{\mathrm{sum}}(k;\mathbb{G}_m,\dots,\mathbb{G}_m)\simeq K^M_r(k)$. Thus the element $\{a,b_1,\dots,b_r\}_k$ is also 2-torsion in $K^{\mathrm{geo}}_{\mathrm{sum}}(k;\mathbb{G}_a,\mathbb{G}_m\dots,\mathbb{G}_m)$ for any $a\in k$. Thus we have

$$\{a,b_1,\dots,b_r\}_k = 2\{\frac{a}{2},b_1,\dots,b_r\}_k = 0.$$

□

4.3. K-group of $\mathbb{G}_a$ and $\mathbb{G}_a$.

Proposition 4.8 (Corollary 5.4)**.** *Assume that $k=\mathbb{C}$. Then the group $K^{\mathrm{geo}}_{\mathrm{sum}}(\mathbb{C};\mathbb{G}_a,\mathbb{G}_a)$ is non-zero.*

This proposition gives a different phenomenon of our K-group $K^{\mathrm{geo}}_{\mathrm{sum}}$ from T-functor (see [5, Theorem 5.5.1]). The proof of Proposition 4.8 will be given in the next section (see Corollary 5.4).

On the other hand, for K-group $K^{\mathrm{geo}}_{\mathrm{max}}$ with maximal modulus conditions for $\mathbb{G}_a$ and $\mathbb{G}_a$, by the proof of [5, Theorem 5.5.1], we have

Proposition 4.9. *Assume that $ch(k)\neq 2$. Then for any $F_i\in$ **RSC**, we have*

$$K^{\mathrm{geo}}_{\mathrm{max}}(k;\mathbb{G}_a,\mathbb{G}_a,F_1,\dots,F_r)=0.$$

Proof. It is enough to show that $\{a,b,c_1,\dots,c_r\}=0$ in $K^{\mathrm{geo}}_{\mathrm{max}}$ for any $a,b\in k$ and any $c_i\in F_i(k)$.

We consider the following functions

$$
\begin{aligned}
g_1 &= \frac{at}{2}\in\mathbb{G}_a(\mathbb{P}^1\setminus\{\infty\})\\
g_2 &= bt\in\mathbb{G}_a(\mathbb{P}^1\setminus\{\infty\})\\
h_i &= c_i\in F_i(\mathbb{P}^1)\quad (i=1,\dots,r)
\end{aligned}
$$

and the rational function f on $\mathbb{P}^1$,

$$f=\frac{t^2-1}{t^2}.$$

Then one can easily check that g_1,g_2 and h_i have SC-modulus $(\mathbb{P}^1,2[\infty]),(\mathbb{P}^1,2[\infty])$ and $(\mathbb{P}^1,)$ respectively, and that f belongs to $G(\mathbb{P}^1,2[\infty])$. Therefore a collection $((\mathbb{P}^1,2[\infty]),f,\{g_1,g_2,h_1,\dots,h_r\})$ is a relation datum of geometric

type with *maximal modulus* for $\mathbb{G}_a, \mathbb{G}_a, F_i$. Thus the corresponding element is zero in $K^{\mathrm{geo}}_{\mathrm{max}}$. Thus

$$0 = \{\frac{a}{2}, b, c_1, \dots, c_r\} + \{\frac{-a}{2}, -b, c_1, \dots, c_r\} = \{a, b, c_1, \dots, c_r\}.$$

This completes the proof. □

5. Applications

5.1. 0-cycles with modulus.

Corollary 5.1. *Let $M_i := (C_i, \mathfrak{m}_i)$ be a proper modulus pair with smooth curves C_i for $i = 1, 2$. Let J_{M_i} be the generalized Jacobian for M_i. Assume that the open complement C_i^o has a k-rational point. Then we have the following decomposition*

$$\mathrm{CH}_0(M_1 \otimes M_2) \simeq \mathbb{Z} \oplus J_{M_1}(k) \oplus J_{M_2}(k) \oplus K^{\mathrm{geo}}_{\mathrm{sum}}(k; J_{M_1}, J_{M_2}).$$

Proof. By [8, Proposition 9.4.1], we know that $h_0^{\mathrm{Zar}}(M_i) \simeq \mathbb{Z} \oplus J_{M_i}$. Taking the tensor product in **RSC** and taking a value over $\mathrm{Spec}(k)$, we have

$$\mathrm{CH}_0(M_1 \otimes M_2) \simeq \mathbb{Z} \oplus J_{M_1}(k) \oplus J_{M_2}(k) \oplus (J_{M_1} \otimes_{\mathbf{RSC}} J_{M_2})(k).$$

Thus the assertion follows from Theorem 3.1. □

Corollary 5.2. *Assume that the base field k is of characteristic zero. Let $M = (\mathbb{P}^1, 2[\infty]), N = (\mathbb{P}^1, [0] + [\infty])$ be the modulus pairs. There is an isomorphism*

$$\mathrm{CH}_0(M \otimes N^{\otimes r}) \simeq \mathbb{Z} \oplus k \oplus (k^*)^{\oplus r} \oplus \bigoplus_{i=2}^{r} (K_i^M(k))^{c_i} \oplus \bigoplus_{i=1}^{r} (\Omega_k^i)^{\oplus c_i}.$$

Here c_i is the binomial coefficient $\binom{r}{i}$. The isomorphism in case $r = 1$ sends a 0-cycle $[(a, b)]$ of a k-rational point to

$$(1, a, b, a\mathrm{dlog}(b)) \in \mathbb{Z} \oplus k \oplus k^* \oplus \Omega_k^1.$$

Proof. We prove $r = 1$ case only. The assertion follows from Corollary 5.1 and Proposition 4.7. □

Remark 5.3. For an effective divisor $\mathfrak{m}$ on $\mathbb{P}^1$, we know that in characteristic zero, $h_0^{\mathrm{Zar}}(\mathbb{P}^1, \mathfrak{m})$ can be described by $\mathbb{Z}$, $\mathbb{G}_m$ and $\mathbb{G}_a$. Thus, similarly to Corollary 5.2, one can explicitly compute Chow group of 0-cycle with modulus for products of $(\mathbb{P}^1, \mathfrak{m})$ and $N^{\otimes r}$.

The next proposition comes from a discussion with Federico Binda. And an idea of the proof is originally given by him.

Corollary 5.4. *Assume that $k = \mathbb{C}$. Then $(\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_a)(\mathbb{C}) \neq 0$ and $(\mathbb{G}_a \otimes_{\mathbf{RSC}} J_{\overline{C}})(\mathbb{C}) \neq 0$ where $J_{\overline{C}}$ is the Jacobian variety of a projective smooth curve $\overline{C}$.*

Proof. Let $M = (\mathbb{P}^1, 2[\infty])$ and $N = (\overline{C}, \phi)$. We consider modulus pairs $L_1 := M \otimes M$ and $L_2 := M \otimes N$. For these modulus pairs, we have $p_g(\overline{L}_1) = 0$ and $p_g(\overline{L}_2) = 0$. Furthermore,

$$H^2(\overline{L}_1, \mathcal{I}_{L_1^\infty}) \neq 0, \qquad H^2(\overline{L}_2, \mathcal{I}_{L_2^\infty}) \neq 0.$$

The modulus pairs L_1 and L_2 satisfies conditions in [2, Theorem 10.9] and hence $\mathrm{CH}_0(L_1)$ and $\mathrm{CH}_0(L_2)$ are not finite-dimensional, i.e. the Albanese kernel is not zero. Now by Corollary 5.1, we have

$$(\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_a)(\mathbb{C}) = K^{\mathrm{geo}}_{\mathrm{sum}}(\mathbb{C}; \mathbb{G}_a, \mathbb{G}_a) \neq 0,$$
$$(\mathbb{G}_a \otimes_{\mathbf{RSC}} J_{\overline{C}})(\mathbb{C}) = K^{\mathrm{geo}}_{\mathrm{sum}}(\mathbb{C}; \mathbb{G}_a, J_{\overline{C}}) \neq 0.$$

□

Remark 5.5. Compare to [5, Theorem 5.5.1] and [12], Corollary 5.4 tells us that in general our K-group $K^{\mathrm{geo}}_{\mathrm{sum}}$ does not coincides with T-functor group.

Corollary 5.6. *Let $(C, \mathfrak{m}_{\mathrm{red}})$ be a proper modulus pair with $\dim C = 1$. Assume that $ch(k) > 2$ and that the open complement C^o has a k-rational point. Then for any $s \geq 1$, we have a decomposition*

$$\mathrm{CH}_0((\mathbb{P}^1, (s+1)[\infty]) \otimes (C, \mathfrak{m}_{\mathrm{red}})) \simeq \mathbb{Z} \oplus \mathbb{W}_s(k) \oplus J(k).$$

Here J is the Jacobian variety of C^o.

Proof. By the definition of the tensor product $\otimes_{\mathbf{RSC}}$ in $\mathbf{RSC}$, we have a surjection

$$(F \otimes_{\mathbf{PST}} G)(k) \longrightarrow (F \otimes_{\mathbf{RSC}} G)(k)$$

for any $F, G \in \mathbf{RSC}$. Thus the assertion now follows from a vanishing result of Hiranouchi [3, Lemma 3.6] and Corollary 5.1. □

5.2. Higher 0-cycle with modulus.

Proposition 5.7. *Let $M = (X, D)$ be a proper modulus pair. Let $d = \dim X$ and let r be a positive integer. Then there is a surjection*

$$cycl_{K^{\mathrm{geo}}_{\mathrm{sum}}} : K^{\mathrm{geo}}_{\mathrm{sum}}(h_0(M), \mathbb{G}_m, \dots, \mathbb{G}_m) \longrightarrow \mathrm{CH}^{d+r}(M; r).$$

Here $\mathrm{CH}^{d+r}(M; r)$ is Binda–Saito's higher Chow group with modulus ([1]).

Proof. We know the isomorphism

$$k^* \simeq \mathrm{CH}^1(k; 1).$$

Consider the external product

$$\mathrm{CH}_0(M) \otimes (k^*)^{\otimes r} \longrightarrow \mathrm{CH}^{d+r}(M; r).$$

One easily shows that this external product induces a surjection

$$(h_0(M) \overset{M}{\otimes} \mathbb{G}_m \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathbb{G}_m)(k) \longrightarrow \mathrm{CH}^{d+r}(M; r).$$

Thus our task now is to show that the element of the form

$$\sum_{c \in C^o} v_c(f) \mathrm{Tr}_{k(c)/k}(g_1(c) \otimes \cdots \otimes g_n(c)) \tag{5.1}$$

goes zero in $\mathrm{CH}^{d+r}(M; r)$ by the external product map.

In our case, the relation data is given by the following:

(i) C: a proper integral normal curve over k
(ii) α: admissible finite correspondence from $(C, \mathfrak{m}_0)$ to M.
(iii) g_i: elements in $\mathbb{G}_m(C \setminus \mathfrak{m}_0)$ $(i = 1, \dots, r)$
(iv) f: a function in $G(C, \mathfrak{m})$ where $\mathfrak{m} = \mathfrak{m}_0 + r(\mathfrak{m}_0)_{\mathrm{red}}$

We may assume that α is elementary, i.e. it is given by one integral closed subscheme Z in $C^o \times X^o$ where $X^o = X \setminus D$.

Let $\overline{Z}^N$ be the normalization of the closure of Z in $C \times X$ and let $p : \overline{Z}^N \to C$ and $q : \overline{Z} \to X$ be the projection. We may assume that q is finite. Then we have a finite map

$$\varphi := q \times p^* f \times p^* g_1 \times \cdots \times p^* g_r : \overline{Z}^N \longrightarrow X \times (\mathbb{P}^1)^{r+1}.$$

Let W be the closed integral subscheme $\mathrm{Im}(\varphi) \cap (X^o \times (\mathbb{P}^1 \setminus \{1\})^{r+1})$. By condition (ii), (iii) and (iv), we have

$$\varphi^*(D \times (\mathbb{P}^1)^{r+1}) \leq \varphi^*(X \times \{1\} \times (\mathbb{P}^1)^r) \leq \varphi^*(X \times F_1).$$

Thus W belongs to $z^{d+r}(M; r+1)$ and we have

$$\partial(W) = \sum_{p \in C^o} v_p(f)[(Z(p), g_1(p), \ldots, g_r(p))]$$

which is the image of (5.1) by the external product map. This completes the proof. $\square$

5.3. **The tensor product of $\mathbb{G}_a$ and $\mathbb{G}_m$.** Assume that the base field k is of characteristic zero.

Corollary 5.8. *There is an isomorphism of reciprocity Zariski sheaves*

$$\Omega^1 \simeq (\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}}.$$

Proof. For any field K of characteristic zero, by Corollary 5.2, we have

$$\mathrm{CH}_0(M_K \otimes N_K) \simeq \mathbb{Z} \oplus \mathbb{G}_m(K) \oplus \mathbb{G}_a(K) \oplus \Omega^1_K$$

where $M_K = (\mathbb{P}^1_K, 2[\infty]), N = (\mathbb{P}^1_K, [0] + [\infty])$.

On the other hand, if K is the function field $k(X)$ of a smooth scheme X over a field k, then we have

$$h_0(M \otimes N)(k(X)) \simeq \mathrm{CH}_0(M_{k(X)} \otimes N_{k(X)}).$$

Since $h_0^{\mathrm{Zar}}(M \otimes N)$ decompose as

$$h_0^{\mathrm{Zar}}(M \otimes N) \simeq \mathbb{Z} \oplus \mathbb{G}_m \oplus \mathbb{G}_a \oplus (\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}},$$

we have an isomorphism

$$\tau_{k(x)} : \Omega^1_{k(X)} \simeq (\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}}(k(X)), \quad a\mathrm{dlog}(b) \mapsto \{a, b\}$$

for all connected smooth scheme X over k.

Now our task is to construct a map $\Omega^1 \to (\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}}$ which is compatible with the above isomorphism. Since a reciprocity sheaves has global injective property [8, Theorem 6], for any smooth X and any point x in X, we have injections

$$\Omega^1_{\mathcal{O}_{X,x}} \to \Omega^1_{k(X)}, \quad (\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}}(\mathcal{O}_{X,x}) \to (\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}}(k(X)).$$

Thus it is enough to check that the isomorphism $\tau_{k(X)}$ sends $\Omega^1_{\mathcal{O}_{X,x}}$ to $(\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m)_{\mathrm{Zar}}(\mathcal{O}_{X,x})$. But it is easily to see, and the proof completes. $\square$

Remark 5.9. The same strategy in the proof of Corollary 5.8 works for proving the following isomorphisms of reciprocity Zariski sheaves.

(1) $(\mathbb{G}_m^{\otimes_{\mathbf{RSC}} r})_{\mathrm{Zar}} \simeq (\mathbb{G}_m^{\otimes_{\mathbf{HI}} r})_{\mathrm{Zar}}$.
(2) $(\mathbb{G}_a \otimes_{\mathbf{RSC}} \mathbb{G}_m^{\otimes_{\mathbf{RSC}} r})_{\mathrm{Zar}} \simeq \Omega^r$.

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